Abstract

In this paper I show how from the 2-soliton solution of the Sine-Gordon equation I create a new solution of this equation. The new solution is oscillatory, but singular.

Motivation

The Sine-Gordon equation, \( u_{tt} - u_{xx} + \sin u = 0 \) [1,2,5], has a N-soliton formula [5], which describes the interaction of an arbitrary number \( N \) of solitons. These types of oscillatory solutions derived from the N-soliton formula can possibly give a better understanding of singular phenomena that can happen in a system, like rogue (freak) waves for example, where the massive wave front can be understood as a singularity created by an unusual phenomenon (like an earthquake). The Sine-Gordon equation is a non-trivial model of the Field Theory as well. These types of oscillatory solutions could bring a better understanding of unusual phenomena in this field. There is a lot of study ahead, but in this paper I want to bring to attention these types of solutions, which usually bring controversy because of the singularity.

Constructing the soliton-like solutions for the Sine-Gordon equation

The Sine-Gordon equation has the analogue representation:

\[
\begin{align*}
\psi_{tt} & = \psi_{xx} + \sin \psi
\end{align*}
\]

I construct the oscillatory solutions for the Sine-Gordon equation (1) by applying a limiting process to the 2-soliton solution of the Sine-Gordon equation (1) [5]:

\[
\psi(x,t) = -4 \arg(\det(I + V))
\]

where \( I \) is the \( 2 \times 2 \) identity matrix and \( V \) is the \( 2 \times 2 \) matrix with the following entries:

\[
V_{ij} = c_j \exp(2i \lambda_j x - it/(2 \lambda_j))/(\lambda_k + \lambda_j), \quad k, j = 1, 2, \quad \iota = \sqrt{-1}.
\]

The parameters \( \lambda_1, \lambda_2 \) and \( c_1, c_2 \) are complex parameters (they can be assigned to be real as well). In the formula (2) we consider: \( \lambda_1 = 1/\mu - \mu \iota, \quad \lambda_2 = -\lambda_1, \quad c_1 = 2\varepsilon \mu \exp(1/(1/ \mu_1 + \varepsilon) + \alpha \iota \varepsilon / 2), \quad \text{and} \quad c_2 = -2\varepsilon \mu \exp(1/(1/ \mu_1 + \varepsilon) + \alpha \iota \varepsilon / 2). \) Taking Taylor series expansion about \( \varepsilon = 0 \), and taking a limiting process as \( \varepsilon \to 0 \), the formula (2) becomes:

\[
\psi(x,t) = -4 \arctan(\text{num}(x,t)/\text{denom}(x,t))
\]

where:
The solution (3) is an oscillatory solution of (1), and it is a singular solution as well. The singularity is a result of the decaying behavior of the oscillations in space and time. The formula (3) is governed by the real parameters $\mu_i \neq 0$, $p_1 \neq 0$, and $\alpha_i$.

\[
\begin{align*}
\text{num}(x,t) &= 2\mu_1^2 \sinh(p_1) \sin((4x - \mu_1^2 t)/(2\mu_1)) \\
\text{denom}(x,t) &= -4\mu_1 x - \mu_1^3 t - \alpha_1 + 2\mu_1^3 \cosh(p_1) \cos((4x - \mu_1^2 t)/(2\mu_1))
\end{align*}
\]

**Figure:** Oscillatory solution in 3D. The first picture shows the regular part of the solution. The second picture shows the singularity. Parameters used: $\mu_1 = -1$, $p_1 = \ln(2)$, $\alpha_1 = 50$.

Here is a beautiful picture showing oscillations near singularity and a personal picture with a twist (using the described solutions):

![Picture 1](http://www.universaltheory.org/Singularity.html)  
![Personal FotoShopped picture](http://www.universaltheory.org/Singularity.html)

**References**